Exercise 8

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

$$1 - x - e^{-x} = \int_0^x (t - x)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Multiply both sides of the integral equation by -1.

$$x + e^{-x} - 1 = \int_0^x (x - t)u(t)\,dt$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\{x + e^{-x} - 1\} = \mathcal{L}\left\{\int_0^x (x - t)u(t)\,dt\right\}$$

Use the fact that the Laplace transform is linear on the left side and apply the convolution theorem on the right side.

$$\begin{aligned} \mathcal{L}\{x\} + \mathcal{L}\{e^{-x}\} - \mathcal{L}\{1\} &= \mathcal{L}\{x\}U(s) \\ \frac{1}{s^2} + \frac{1}{s+1} - \frac{1}{s} &= \frac{1}{s^2}U(s) \end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned} U(s) &= 1 + \frac{s^2}{s+1} - s \\ &= \frac{1(s+1) + s^2 - s(s+1)}{s+1} \\ &= \frac{1}{s+1} \end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned} u(x) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= e^{-x} \end{aligned}$$